Preaching about random temporal trees

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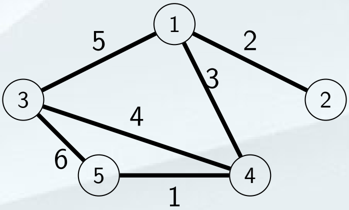
What is a temporal graph?

Definition: Temporal graph

Let $G = (V, E)$ be a graph, and $\lambda : E \to [0, \infty)$ be an injective function. The pair (G, λ) is called a temporal graph.

Definition: Reachability

For vertices $u, v \in G$, we say that u **can reach** v if there exists a path P from u to v where λ increases as we travel from u to v along P.



What is a temporal graph?

Definition: Random simple temporal graph

A temporal graph (G, λ), where G is an Erdős-Rényi random graph and $(\lambda(e) : e \in E)$ is a collection of independent uniform$[0, 1]$ random variables.

- **Network science motivations:** disease spread, information flow on social networks, etc.
- **Mathematical motivations:** This new definition of reachability is not symmetric or transitive, which complicates the analysis of phase transitions.
- In recent years, a lot of effort has been put towards understanding this model.

Towards random temporal trees

Some motivation for temporal trees

- When studying sparse Erdős-Rényi random graphs, approximating the neighbourhood around a vertex with a binomial(n, p) offspring distribution Bienaymé-Galton-Watson tree is a commonly used technique.
- This tree-based approximation has been used to study random simple temporal graphs as well.
- However, outside of the context of random simple temporal graphs, temporal Bienaymé-Galton-Watson trees have not been studied.

Towards random temporal trees

Definition: Uniform temporal tree

Let T_n be an infinite n-ary rooted plane tree, with independent uniform$[0,1]$ random variables, U_e, assigned to each edge. Let $\mathcal{T}_{n,p}$ be obtained from T_n by deleting all vertices who's unique path from the root to it is not strictly **decreasing**, with all edge labels less than p. We call $\mathcal{T}_{n,p}$ a uniform temporal tree.

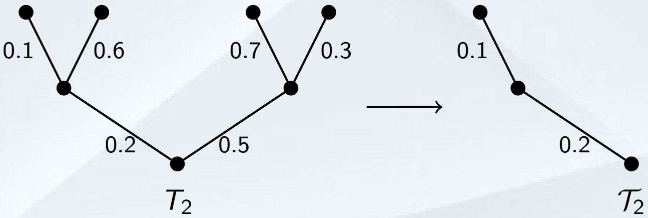


Figure: The first three generations in a realization of $\mathcal{T}_{2,0.4}$.

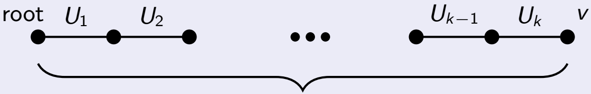
Results on uniform temporal trees

Proposition (A., Devroye, Lugosi 2025+)

For all $n \geq 1$, $\mathbf{E}|\mathcal{T}_{n,p}| = e^{np}$.

Proof

- There are n^k vertices in the kth generation of T_n.
- Each of the vertices in the kth generation are in $\mathcal{T}_{n,p}$ with probability $\frac{p^k}{k!}$.



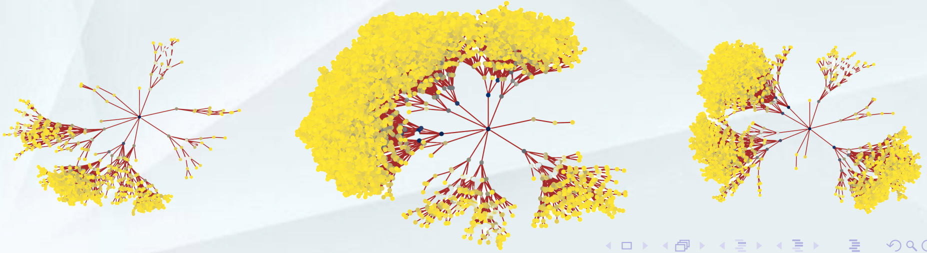
$$\mathbf{P}(v \in \mathcal{T}_n) = \mathbf{P}(p \geq U_1 \geq U_2 \geq \ldots \geq U_k) = \frac{p^k}{k!}$$

- $\mathbf{E}|\mathcal{T}_{n,p}| = \sum_{k=0}^{\infty} \frac{(np)^k}{k!} = e^{np}$.

Results on uniform temporal trees

Theorem (A., Devroye, Lugosi 2025+)

- $\frac{|\mathcal{T}_{n,p}|}{e^{np}} \xrightarrow{\mathcal{L}} E$, where E is an exponential(1) random variable.
- Let H_n be the height of $\mathcal{T}_{n,p}$. Then, $\frac{H_n}{np} \xrightarrow{\mathbb{P}} e$.
- and some more...



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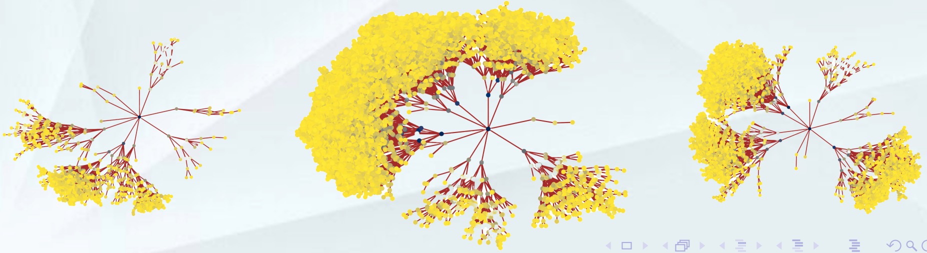
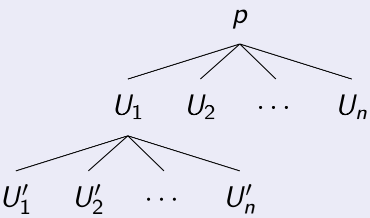


Initial exploration

The first two generations of $\mathcal{T}_{n,p}$

- Give the root label p, and give every other vertex the label of its incoming edge.
- In the first generation, vertices with label above p are deleted.
- Below the leftmost child (if it isn't deleted in the step above), vertices with label above U_1 are deleted.



Initial exploration

How does the tree evolve?

- Give the root label p, and give every other vertex the label of its incoming edge.
- In the first generation, vertices with label above p are deleted.
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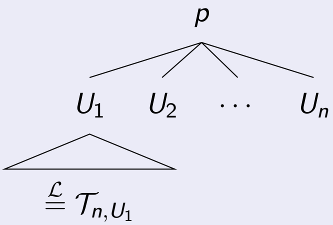
Subtrees below a vertex with label ℓ are distributed like $\mathcal{T}_{n,\ell}$!



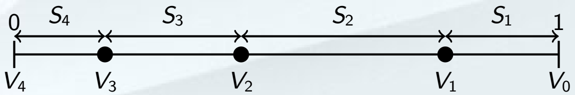
Some facts about uniform spacings

Lemma

- Let $U_1, \ldots, U_n$ be a collection of independent uniform$[0, 1]$ random variables. Set $V_0 = 1$, $V_{n+1} = 0$, and let $V_1 \geq \ldots \geq V_n$ be $U_1, \ldots, U_n$ re-ordered from greatest to least.
- Define $S_k = V_{k-1} - V_k$ for all $k \in \{1, \ldots, n+1\}$.
- Let $(E_k)_{k=0}^{\infty}$ be a sequence of independent exponential(1) random variables.

Then,

$$(S_1, \ldots, S_{n+1}) \stackrel{\mathcal{L}}{=} \left(\frac{E_1}{\sum_{k=1}^{n+1} E_k}, \ldots, \frac{E_{n+1}}{\sum_{k=1}^{n+1} E_k} \right).$$



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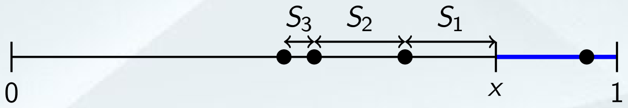
- **Note:** For any fixed $L > 0$, $n \cdot (S_1, ..., S_L) \xrightarrow{\mathcal{L}} (E_1, ..., E_L)$ as $n \to \infty$.

Some facts about uniform spacings

More spacings

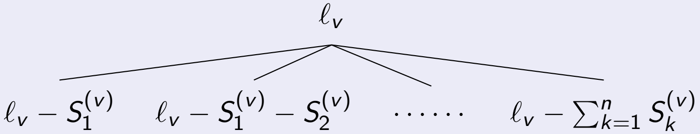
- When looking at $\mathcal{T}_{n,p}$, we only kept the vertices with label below the label of its parent.
- When we only look at entries in a vector of uniforms $(U_1, \ldots, U_n)$ that are below a fixed $x \in (0, 1)$, the gaps are still distributed like uniform spacings.



The uniform spacings coupling

A new construction of uniform temporal trees

- For each $v \in T_n$, associate a vector of uniform spacings $(S_1^{(v)}, ..., S_{n+1}^{(v)})$.
- We define labels for each vertex $(\ell_v : v \in T_n)$. We start with the root having label p, and the rest are defined recursively according to the following picture:



- Finally, let $\mathcal{T}_{n,p}$ be obtained by deleting all vertices that have negative label.

The uniform spacings coupling

A new (approximate) construction of uniform temporal trees

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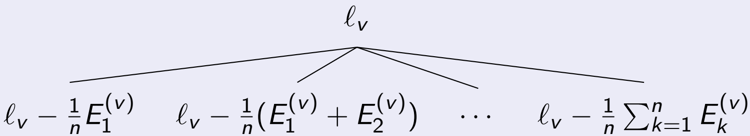

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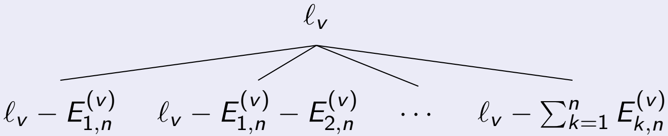

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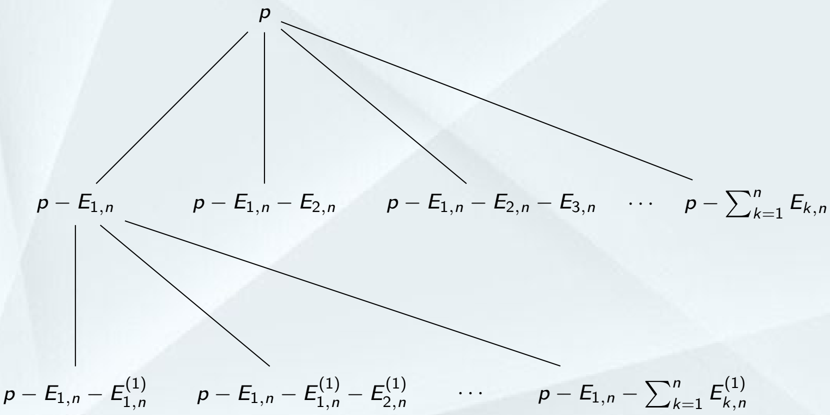
- We define labels for each vertex $(\ell_v : v \in T_n)$. We start with the root having label p, and the rest are defined recursively according to the following picture:

$$\ell_v$$

$$\ell_v - E_{1,n}^{(v)} \qquad \ell_v - E_{1,n}^{(v)} - E_{2,n}^{(v)} \qquad \cdots \qquad \ell_v - \sum_{k=1}^{n} E_{k,n}^{(v)}$$

- Finally, let $\mathcal{T}_{n,p}$ be obtained by deleting all vertices that have negative label.
- We call the leftmost child the rank 1 child of v, the second to the left the rank 2, and so on...

The uniform spacings coupling



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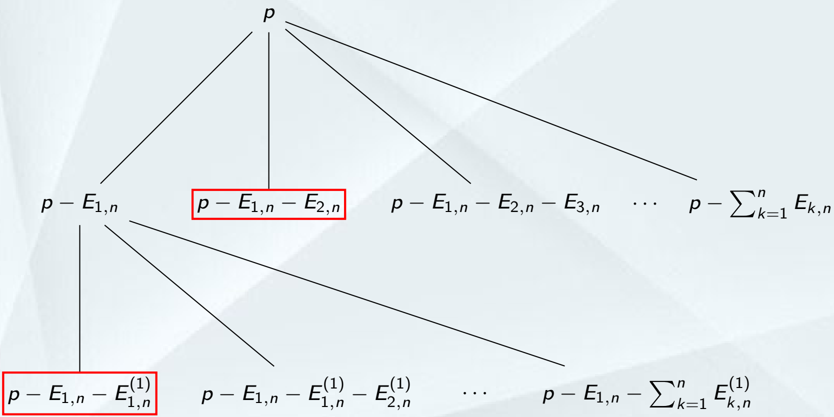

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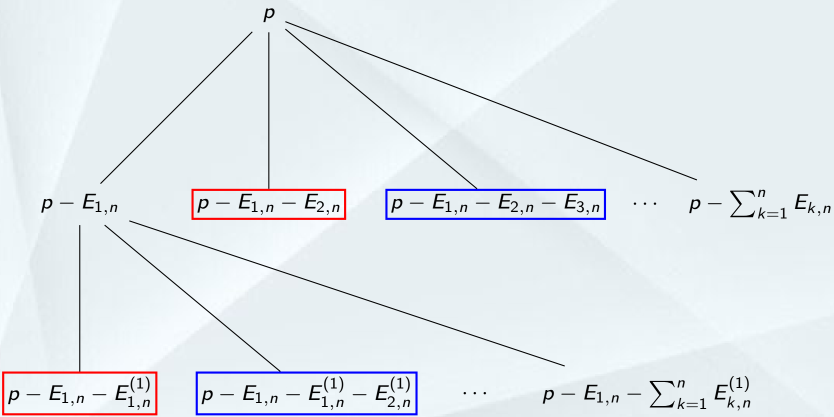

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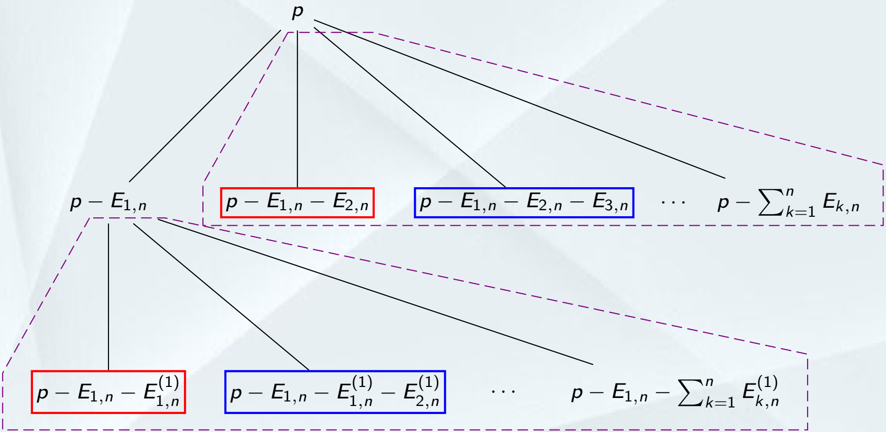
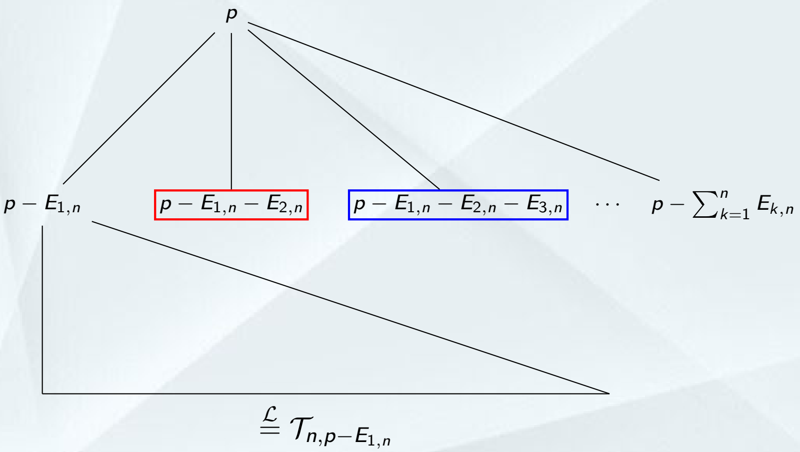


- Notice that the two portions are conditionally independent given the label of the leftmost child of the root.

The uniform spacings coupling



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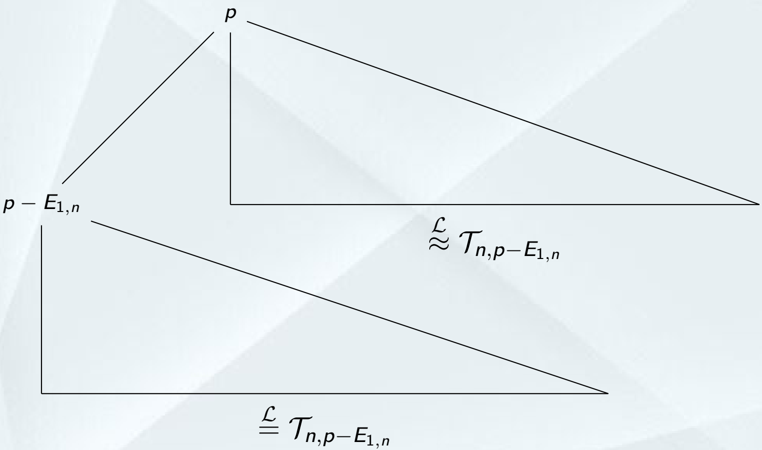
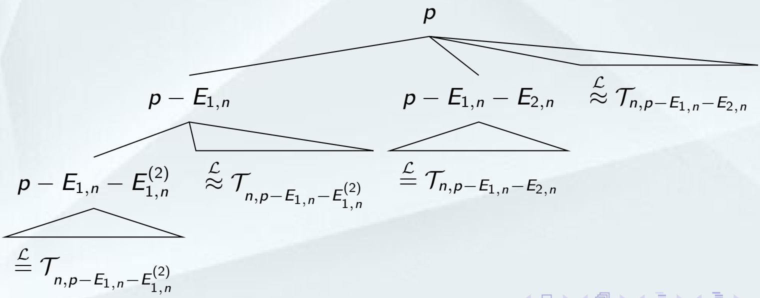


- The two subtrees are (approximately) identically distributed, and conditionally independent given the label of the leftmost child.

The hidden branching random walk

Transforming T_n into a binary tree

- From T_n we construct a new tree T_n^b according to the following rule:
- Let $v \in T_n$. In T_n^b, the left child of v is its child of largest rank in T_n, and the right child is the sibling of v in T_n of rank one higher (if one exists).



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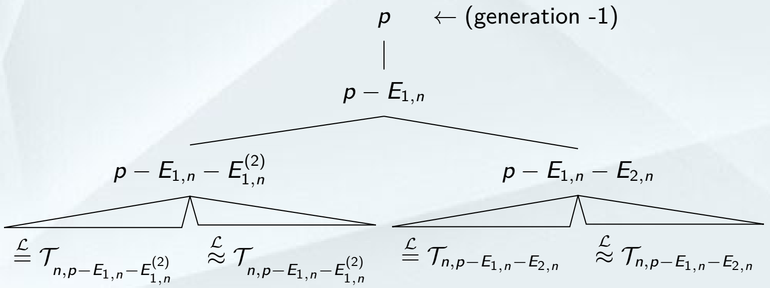
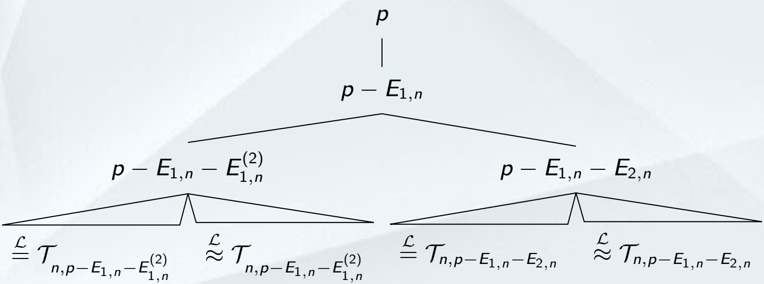


The hidden branching random walk

Transforming T_n into a binary tree

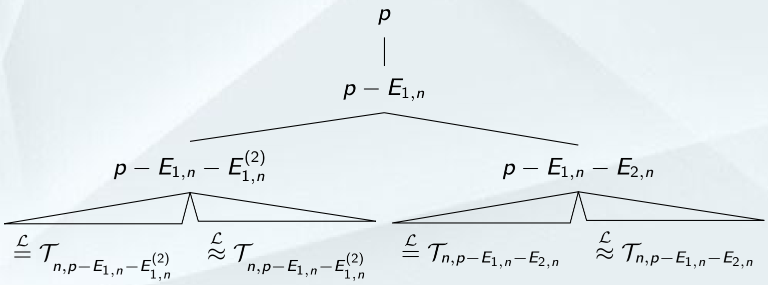
- The vertex labels in T_n^b evolve according to a branching random walk with step size $\frac{1}{n}E$, where $E \overset{\mathcal{L}}{=}$ exponential(1).
- The subtrees hanging below a generation L are all conditionally independent given the labels in generation L.



The hidden branching random walk

Transforming T_n into a binary tree

- The subtrees hanging below a generation L are all conditionally independent given the labels in generation L. $\rightarrow$ Note that, for a **fixed** $L > 0$, all vertices in generation L have positive label with high probability. Moreover, up to generation L, T_n^b contains a bounded number of vertices.



Putting it all together

Conclusions from the binary tree conversion

- Let $v_1, \ldots, v_{2^L}$ be the vertices in a fixed generation L of T_n^b.
- Let $(X_v : v \in T_n^b)$ be branching random walk on T_n^b with step size exponential(1).
- Let $\mathcal{T}_1(v_i)$ be the left subtree of v_i in T_n^b, and $\mathcal{T}_2(v_i)$ the right subtree **after we delete negative label vertices**.

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- Following from the remarks from the last slide, we know that with high probability

$$|\mathcal{T}_{n,p}| \sim \sum_{i=1}^{2^L} \Big(|\mathcal{T}_1(v_i)| + |\mathcal{T}_2(v_i)| \Big).$$

Putting it all together

Conclusions from the binary tree conversion

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$$|\mathcal{T}_{n,p}| \sim \sum_{i=1}^{2^L} \Big(|\mathcal{T}_1(v_i)| + |\mathcal{T}_2(v_i)| \Big).$$

- One can use the conditional independence of the $\mathcal{T}_i(v_j)$'s to argue that the above sum really behaves like

$$\sum_{i=1}^{2^L} \mathbf{E}\Big[|\mathcal{T}_1(v_i)| + |\mathcal{T}_2(v_i)| \ \Big| \ (\ell_{v_1}, ..., \ell_{v_{2^L}}) \Big].$$

Putting it all together

Conclusions from the binary tree conversion

Using the BRW connection, the labels of the vertices in the Lth generation satisfy $(\ell_{v_1}, ..., \ell_{v_{2^L}}) \stackrel{\mathcal{L}}{=} p - n^{-1}(X_{v_1}, ..., X_{v_{2^L}})$.

Putting it all together

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$$
\begin{aligned}
|\mathcal{T}_{n,p}| &\approx \sum_{i=1}^{2^L} \mathbf{E}\Big[|\mathcal{T}_1(v_i)| + |\mathcal{T}_2(v_i)| \;\Big|\; (\ell_{v_1}, \dots, \ell_{v_{2^L}})\Big] \\
&\approx 2\sum_{i=1}^{2^L} \mathbf{E}\Big[|\mathcal{T}_1(v_i)| \;\Big|\; \ell_{v_i}\Big] = 2\sum_{i=1}^{2^L} e^{n(p-n^{-1}X_{v_i})} = 2\sum_{i=1}^{2^L} e^{np-X_{v_i}}.
\end{aligned}
$$

Putting it all together

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- $|\mathcal{T}_{n,p}|/e^{np} \approx 2\sum_{i=1}^{2^L} e^{-X_{v_i}} := 2X_L$.

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$$

- $|\mathcal{T}_{n,p}|/e^{np} \approx 2 \sum_{i=1}^{2^L} e^{-X_{v_i}} := 2X_L$.
- X_L is a martingale, and so has some limit X. Using the recursive properties of X_L we can compute the moments of X and show that that $X \stackrel{\mathcal{L}}{=} \frac{1}{2}\text{exponential}(1)$.

Thank you all for listening :)

- The QR code below leads to some references for papers on random temporal graphs (These slides are on my website too)! There are plenty of cool open problems surrounding these topics - come ask me about them!

(a) QR Code

(b) Mathematicians